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COMPRESSIBILITY EFFECTS ON WAVES IN STRATIFIED TWO-PHASE FLOW

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Abstract—Ardron (1980) presented both one-dimensional and two-dimensional analyses of wave propagation in horizontal stratified two-phase flow. He compared the two approaches and concluded that the comparison helped to improve confidence in the use of one-dimensional approximations for the analysis of complex systems such as nuclear reactors.

There are several assumptions in Ardron's developments. When alternative assumptions are made the results change. By examining the consequences of several possible assumptions we have learned from this example that considerable care may be necessary in the reduction of a multi-dimensional two-phase flow problem to a simpler form.

This paper presents a more complete two-dimensional solution of this problem and discusses the limitations of the approximate solutions.

INTRODUCTION

A current topic of debate in the technical community is the degree to which two-phase flow phenomena, that are usually multidimensional in character, can be modeled by using a one-dimensional approach. A good way to illustrate the features of this problem would seem to be to obtain both complete and approximate solutions to some relatively well-defined situations and examine the conditions under which the former can be reduced to the latter.

One of the first significant attempts to work out a complete example on these lines is the recent paper by Ardron (1980). He reviews some of the background literature and applications and develops solutions for the case of wave propagation in the horizontal stratified flow of a gas over a liquid.

In studying Ardron's paper we were puzzled by a few features that led us to extend his approach and, in some instances, develop alternative assumptions or more complete derivations. The main contributions of the present paper are:

(1) Demonstrating that there are many possible alternative assumptions that can be made to develop *ad hoc* one-dimensional constitutive equations, and that these lead to different dispersion relationships for the waves.

(2) Development of a more complete two-dimensional solution that incorporates a class of vertical density variations in the steady perturbation variables without the need for the introduction of specific assumptions about the ratio between the speeds of compressibility and gravity waves.

(3) Averaging of the two-dimensional solution to obtain the proper form of the variables for use in the one-dimensional model. These variables satisfy the one-dimensional conservation equations but are not generally identical with the same variables derived from ad hoc assumptions.

(4) Clarification of several aspects of Ardron's two-dimensional solution. Since our approaches are similar to Ardron's, except for the details, we have used his nomenclature throughout (even to the extent of adopting the nomenclature of his appendix, that differs slightly from the body of his paper, in the appendix to this paper). A few additional symbols are explained when they are introduced.

ARDRON'S ONE-DIMENSIONAL SOLUTION

Ardron starts from one-dimensional forms of the equations of conservation of mass and momentum for two stratified fluids with uniform properties and no heat or mass transfer:

$$\frac{\partial}{\partial t}(\alpha_j\bar{\rho}_j) + \frac{\partial}{\partial x}(\alpha_j\bar{\rho}_j\bar{U}_j) = 0 \quad [1]$$

$$\alpha_j\bar{\rho}_j\frac{\partial \bar{U}_j}{\partial t} + \alpha_j\bar{\rho}_j\bar{U}_j\frac{\partial \bar{U}_j}{\partial x} + \alpha_j\frac{\partial \bar{P}_j}{\partial x} - (P^* - \bar{P}_j)\frac{\partial \alpha_j}{\partial x} = 0.. \quad [2]$$

He relates the interface pressure P^* to the average pressures in each phase by

$$P^* = \bar{P}_G + 1/2\alpha_G\rho_G gH = \bar{P}_L - 1/2\alpha_L\rho_L gH \quad [3]$$

and uses the speeds of sound to relate changes in the mean density to changes in the mean pressure:

$$\hat{\bar{\rho}}_j = \hat{\bar{P}}_j/c_j^2. \quad [4]$$

He substitutes for $\bar{P}_j$ in [2] using[3], neglecting the x-derivatives of ρ_G and ρ_L, although[4] shows that they should be related to the derivatives of pressure. His resulting equation[3], when the common factor α_j is removed, is

$$\bar{\rho}_j\frac{\partial \bar{U}_j}{\partial t} + \bar{\rho}_j\bar{U}_j\frac{\partial \bar{U}_j}{\partial x} + \frac{\partial P^*}{\partial x} \pm \bar{\rho}_j gH\frac{\partial \alpha_j}{\partial x} = 0 \quad [5]$$

where the positive sign applies to the equation for the liquid phase.

Ardron uses a perturbation technique to solve[1], [4], and [5] for fluctuations in P^*, ρ_j, α_j and $\bar{U}_j$. Since P^* does not appear in [4] he must have made some assumption or substitution that is not explained in his paper. We have concluded that he probably made the assumption:

$$\hat{P}^* \simeq \hat{\bar{P}}_j \quad [6]$$

in order to derive his dispersion relation that we prefer to write in the more obviously symmetrical form:

$$\frac{\rho_G\omega_G^2}{\alpha_G\mu_G^2} + \frac{\rho_L\omega_L^2}{\alpha_L\mu_L^2} = gHk^2\left(\frac{\rho_L}{\mu_L^2} - \frac{\rho_G}{\mu_G^2}\right) \quad [7]$$

with $\omega_j = \omega - U_{j0}k$ and $\mu_j^2 = (k^2 - \omega_j^2/c_j^2)$.

In essence, Ardron solves the problem in which the phase pressure fluctuations are regarded as equal, with gravity appearing only in the additional "force due to void fraction gradient" that appears in [5], much as it appears in the example in Wallis (1969). Indeed, [7] can be obtained from [6.102] in Wallis (1969) by putting $f_{1v_\alpha} = \rho_1 gH$, $f_{2v_\alpha} = \rho_2 gH$, $v_1' = v_1 - c$, $v_2' = v_2 - c$. This is an interesting case but it is not clear that the assumptions made to reduce the equations to this particular form are appropriate for the stratified flow problem under consideration. The basic problem is how to relate the effective forces acting on the phases to the variations in mean density and void fraction—i.e. to deduce the "constitutive equations".

ONE-DIMENSIONAL CONSTITUTIVE RELATIONS

Since [4] relates density changes to the actual average phase pressures, $\bar{P}_j$, it seems unnecessary to introduce P^* into the equation set. Accordingly, we prefer to use [3] to substitute for the final term in [2] and obtain

$$\alpha_i\bar{\rho}_i\frac{\partial \bar{U}_i}{\partial t}+\alpha_i\bar{\rho}_i\bar{U}_i\frac{\partial \bar{U}_i}{\partial x}+\alpha_i\frac{\partial \bar{P}_i}{\partial x}\pm 1/2\alpha_i\bar{\rho}_i gH\frac{\partial \alpha_i}{\partial x}=0. \quad [8]$$

The factor α_i can be removed from [8] by division.

There are eight unknowns in [1], [4], and [8]. In order to obtain a set that can be solved we need to use the kinematic constraint:

$$\alpha_G+\alpha_L=1 \quad [9]$$

as well as some form of constitutive relationships that relate "forces", in the form of $\hat{P}_G$, $\hat{P}_L$, $\hat{P}^*$, $\rho_i gH\hat{\alpha}_i$, to changes in density (or concentration) in the form of $\hat{\rho}_G$, $\hat{\rho}_L$ and $\hat{\alpha}_i$. This is the point at which the solutions diverge because many alternative assumptions can be made about these relationships, in the absence of a more complete theory based on the multidimensional solution.

For example, instead of using [6] we can use the complete perturbation of [3] in which α_L, α_G, ρ_L and ρ_G are all allowed to vary. It could also be realized that the densities ρ_i that appear in [3] are not the average densities across the entire flow but correspond to a mean density between the interface and some average point in the flow. Another approach is to assume that $\hat{\bar{P}}_i$ is not equal to the perturbation in interface pressure but instead is equal to the perturbed pressure along some "typical average streamline" in the flow, which could be, for example, the top and bottom of the channel, or the mid points of each phase.

In this way we have derived many different forms of dispersion relationships, all of which resemble [7] but contain additional terms or factors on both sides of the equation. As might be expected, these factors often take the form $(1\pm ngH\alpha_i/c_i^2)$, where n is a number, and indicate that we are dealing with corrections of order gH/c_i^2. However, it is not adequate to replace these factors by unity and all of the expressions do not reduce to [7] to first order in gH/c_i^2. We will not repeat all these solutions in this paper (they should be available in a thesis by the second author) but it is worthwhile to indicate one of the features that emerges. When a term on the l.h.s. of [7] is multiplied by a correction factor such as $(1+ngH\alpha_i/c_i^2)$ the second part of the expanded term is

$$ngH\frac{\rho_i\omega_i^2}{\mu_i^2c_i^2}=ngH\rho_i\left(\frac{k^2}{\mu_i^2}-1\right)$$

which is comparable with terms on the r.h.s. of [7] and should be retained.

Since the main theme of this paper is the inadequacy of one-dimensional approaches to this problem (resulting from the fact that transverse compressibility waves must exist under almost all conditions in at least one of the phases) we will not describe these alternative solutions further. It will suffice to show that at least one "reasonable" solution, namely [7], is not compatible with the reduction of our two-dimensional solution to one-dimensional form.

THE TWO-DIMENSIONAL SOLUTION

We seek solutions to the problem posed by Ardron but avoid making any simplifying assumptions until the "general solution" has been obtained. We start with Ardron's equations [9] and perturb them without any change in coordinate system. Moreover, we allow there to be density gradients $\partial\rho_0/\partial y$ in the vertical direction in the unperturbed state, since it is unlikely that vertical compressibility effects can be ignored if H is large enough for gravitational effects to interact with horizontal compressibility effects.

The continuity equation becomes, to first order in perturbations,

$$\frac{\partial \hat{\rho}_i}{\partial t} + \rho_{0i}\left(\frac{\partial \hat{u}_i}{\partial x} + \frac{\partial \hat{v}_i}{\partial y}\right) + U_{0i}\frac{\partial \hat{\rho}_i}{\partial x} + \hat{v}_i\frac{\partial \rho_{0i}}{\partial y} = 0 \qquad [10]$$

while the x- and y-components of momentum conservation are

$$\rho_{0i}\frac{\partial \hat{u}_i}{\partial t} + \rho_{0i}U_{0i}\frac{\partial \hat{u}_i}{\partial x} = -\frac{\partial \hat{P}_i}{\partial x} \qquad [11]$$

$$\rho_{0i}\frac{\partial \hat{v}_i}{\partial t} + \rho_{0i}U_{0i}\frac{\partial \hat{v}_i}{\partial x} = -\frac{\partial \hat{P}_i}{\partial y} - \hat{\rho}_i g \qquad [12]$$

Using $\hat{\rho}_i = \hat{P}_i/c_i^2$ to relate density and pressure fluctuations, we seek a solution in which $c_i^2 = \text{constant}$ – which would normally require that each phase be isothermal. Under this condition the ideal gas law yields

$$\frac{1}{\rho_{0i}}\frac{\partial \rho_{0i}}{\partial y} = \frac{1}{\rho_{0i}RT_i}\left(\frac{\partial P_{0i}}{\partial y}\right) = \frac{1}{\rho_{0i}RT_i}(\rho_{0i}g) = \frac{g}{RT_i} = a_i. \qquad [13]$$

(For the mathematical developments that follow it is not necessary that the phases be isothermal perfect gases, merely that c_i and $(\partial\rho_{0i}/\partial y)/\rho_{0i}$ be constant).

Using the plane wave perturbation the conservation equations now reduce, in terms of the constant a_i, to

$$P_i' = \left(\frac{\rho_{0i}\omega_i}{k}\right)u_i' \qquad [14]$$

$$v_i' = \frac{i}{k}\left(a_i + \frac{g}{c_i^2} + \frac{\partial}{\partial y}\right)u_i' \qquad [15]$$

$$\left[\left(a_i^2 + \frac{a_i g}{c_i^2} - \mu_i^2\right) + \left(2a_i + \frac{g}{c_i^2}\right)\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right]u_i' = 0. \qquad [16]$$

The solution of this second order homogeneous equation [16] is

$$u_i' = A_i e^{\lambda_{1i}y} + B_i e^{\lambda_{2i}y} = e^{b_i y}(A_i e^{\lambda_i y} + B_i e^{-\lambda_i y}) \qquad [17]$$

with:

$$\lambda_{1i}, \lambda_{2i} = -\left(a_i + \frac{g}{2c_i^2}\right) \pm \left(\frac{g^2}{4c_i^4} + \mu_i^2\right)^{1/2} = b_i \pm \lambda_i \qquad [18]$$

Using [17] the solutions [14] and [15] become

$$p_i' = \left(\frac{\rho_{0i}\omega_i}{k}\right)e^{b_i y}(A_i e^{\lambda_i y} + B_i e^{-\lambda_i y}) \qquad [19]$$

$$v_i' = \frac{i}{k}e^{b_i y}\left[\left(\frac{g}{2c_i^2} + \lambda_i\right)A_i e^{\lambda_i y} + \left(\frac{g}{2c_i^2} - \lambda_i\right)B_i e^{-\lambda_i y}\right]. \qquad [20]$$

The perturbations thus have a y-dependence that consists of an exponential decay (or growth) and a trigonometric or hyperbolic term.

The constants A_j and B_j are determined by application of the boundary conditions, as follows. The vertical fluctuation velocities must vanish at the walls; therefore:

$$v_G' = 0 \text{ at } y = h_G; \qquad v_L' = 0 \text{ at } y = -h_L.$$

Also they must be compatible with the motion of the interface, $\hat{\eta} = \eta' \exp i(\omega t - kx)$, and so we require

$$v_j'\bigg|_{y=\hat{\eta}} = \frac{\partial \hat{\eta}}{\partial t} + U_{0j}\frac{\partial \hat{\eta}}{\partial x} \text{ or } v_j' = i\omega_j \eta'$$

where to first order in perturbations we may evaluate v_j' at $y = 0$.

Using [19] and [20] these three conditions become

$$\left(\frac{g}{2c_G^2} + \lambda_G\right) A_G\, e^{\lambda_G h_G} + \left(\frac{g}{2c_G^2} - \lambda_G\right) B_G\, e^{-\lambda_G h_G} = 0 \qquad [21]$$

$$\left(\frac{g}{2c_L^2} + \lambda_L\right) A_L\, e^{-\lambda_L h_L} + \left(\frac{g}{2c_L^2} - \lambda_L\right) B_L\, e^{\lambda_L h_L} = 0 \qquad [22]$$

$$\eta' = \frac{1}{k\omega_G}\left[\left(\frac{g}{2c_G^2} + \lambda_G\right) A_G + \left(\frac{g}{2c_G^2} - \lambda_G\right) B_G\right] =$$

$$= \frac{1}{k\omega_L}\left[\left(\frac{g}{2c_L^2} + \lambda_L\right) A_L + \left(\frac{g}{2c_L^2} - \lambda_L\right) B_L\right]. \qquad [23]$$

The final boundary condition is that the difference in pressures at the interface be balanced by surface tension, σ:

$$(P_{G0} + \hat{P}_G)\bigg|_{y=\hat{\eta}} - (P_{L0} + \hat{P}_L)\bigg|_{y=\hat{\eta}} = \sigma \frac{\partial^2 \hat{\eta}}{\partial x^2}. \qquad [24]$$

Assuming unperturbed hydrostatic pressures $P_{0j}(y) = P_0^* - \rho_j g y$ near $y = 0$ and evaluating the perturbed pressures [19] at $y = 0$ (a first order approximation that can be justified by expanding to higher order terms in $\hat{\eta}$), [24] becomes

$$\eta' g(\rho_L - \rho_G) + \frac{\rho_G \omega_G}{k}(A_G + B_G) - \frac{\rho_L \omega_L}{k}(A_L + B_L) = -\sigma k^2 \eta'. \qquad [25]$$

The boundary conditions [21], [22], [23] and [25] form a homogeneous linear equation set for the constants A_j and B_j. When the system determinant is set equal to zero the resulting dispersion relation is

$$\frac{\rho_G \omega_G^2}{\mu_G^2}\lambda_G \coth(\lambda_G h_G) + \frac{\rho_L \omega_L^2}{\mu_L^2}\lambda_L \coth(\lambda_L h_L)$$

$$= g\left[\rho_L\left(1 + \frac{\omega_L^2}{2\mu_L^2 c_L^2}\right) - \rho_G\left(1 + \frac{\omega_G^2}{2\mu_G^2 c_G^2}\right)\right] + \sigma k^2. \qquad [26]$$

Using the definitions of ω_j and μ_j the r.h.s. of [26] can be rearranged to

$$\frac{g}{2}\left[\rho_L\left(1+\frac{k^2}{\mu_L^2}\right)-\rho_G\left(1+\frac{k^2}{\mu_G^2}\right)\right]+\sigma k^2$$

DERIVATION OF ONE-DIMENSIONAL AVERAGES OF THE TWO-DIMENSIONAL VARIABLES

If we take the first term in the expansion of the hyperbolic cotangent terms in [26] (not strictly a "long wave" approximation since, in addition to kh_j being small, we must also have $\omega_j h_j/c_j \ll 1$ and $gh_j/c_j^2 \ll 1$) we obtain the l.h.s. of [7]. However, the gravitational terms on the r.h.s. are not compatible. This indicates that something has been left out of the one-dimensional model that it may be impossible to include without resorting to some approximate treatment of the y-variations in the flow, as was done in a simpler incompressible flow example by Banerjee (1980).

We may assess the errors introduced in approximation by using the two-dimensional solution to derive the true expressions for the "averages" that appear in equations [1] to [4]. Since we are going to use a perturbation technique we also need to compute the "perturbations of the averages". For a general variable defined as in Ardron's [4],

$$\psi=\psi_0+\psi'\,\mathrm{e}^{i(\omega t-kx)} \qquad [27]$$

we are interested in

$$\bar{\psi}_j'\,\mathrm{e}^{i(\omega t-kx)}=\bar{\psi}_j-\bar{\psi}_{0j}=\frac{\int_\eta^{h_j}\psi\,\mathrm{d}y}{h_j-\eta}-\frac{\int_0^{h_j}\psi_{0j}\,\mathrm{d}y}{h_j} \qquad [28]$$

which to first order in $\eta=\eta'\,\mathrm{e}^{i(\omega t-kx)}$, becomes

$$\bar{\psi}_j'=\frac{\int_0^{h_j}\psi_j'\,\mathrm{d}y}{h_j}+\left(\frac{\int_0^{h_j}\psi_{0j}\,\mathrm{d}y}{h_j}-\psi_0^*\right)\frac{\eta'}{h_j} \qquad [29]$$

where ψ_0^* is the value of ψ_0 at the two-phase interface, $y=0$. The second term in [29] is zero for all variables that are uniform in the unperturbed state but it cannot be neglected when ψ_0 is a function of y, as is the case with the pressure.

Using [21] to [23] to determine A_j and B_j and substituting in [17] we obtain, using the definition of λ_j in [18],

$$\frac{u_j'}{\eta'}=\frac{\mathrm{e}^{b_j y}k\omega_j}{\mu_j^2}\left[\frac{\mathrm{e}^{\lambda_j y}\left(\lambda_j-\frac{g}{2c_j^2}\right)+\mathrm{e}^{-\lambda_j y}\,\mathrm{e}^{2\lambda_j h_j}\left(\lambda_j+\frac{g}{2c_j^2}\right)}{1-\mathrm{e}^{2\lambda_j h_j}}\right]. \qquad [30]$$

At $y=0$ this reduces to

$$\frac{u_j'}{\eta'}=\frac{k\omega_j}{\mu_j^2}\left(\mp\lambda_j\coth\lambda_j h_j-\frac{g}{2c_j^2}\right) \qquad [31]$$

where the upper sign refers to the lighter phase.

Using [31] in [14] and realizing, as in [24], that the perturbation at the interface is the sum of the "in place" perturbation and the change due to the motion of the interface in the "unperturbed" field we obtain the pressures along the interface as

$$P_G'^* = -\rho_{0G} g\eta' - \rho_{0G}\frac{\omega_G^2}{\mu_G^2}\left(\frac{g}{2c_G^2} + \lambda_G \coth \lambda_G h_G\right)\eta' \qquad [32]$$

$$P_L'^* = -\rho_{0L} g\eta' - \rho_{0L}\frac{\omega_L^2}{\mu_L^2}\left(\frac{g}{2c_L^2} - \lambda_L \coth \lambda_L h_L\right)\eta' \qquad [33]$$

from which the dispersion relation [26] follows.

To obtain the perturbations in the averaged "one-dimensional" variables we use [29], substituting for u_i' from [30], using [14] to obtain p_i' and deducing the density perturbations from $\rho_i' = p_i'/c_i^2$. The resulting perturbations in the averaged variables are, if $a_i = 0$ (no initial vertical density gradient),

$$\frac{u_i'}{\eta'} = \mp\frac{k\omega_i}{h_i\mu_i^2}. \qquad [34]$$

$$\frac{\bar{P}_i'}{\eta'} = \mp\frac{\rho_{0i}\omega_i^2}{h_i\mu_i^2} - \frac{\rho_{0i}g}{2} \qquad [35]$$

$$\frac{\bar{\rho}_i'}{\eta'} = \mp\frac{\rho_{0i}\omega_i^2}{h_i\mu_i^2 c_i^2} \qquad [36]$$

while the perturbations in volume fraction are:

$$\frac{\alpha_i'}{\eta'} = \mp\frac{1}{H} \qquad [37]$$

The final term in [35] is a consequence of the final term in [29] and is related to the final term in [2]. The existence of this term explains why one cannot derive [2] by merely placing "averaging" signs over the terms in the differential x-direction momentum equation.

It may easily be checked that the parameters in [34]–[37] satisfy identically the plane wave versions of the perturbed forms of [1] and [2], namely

$$\alpha_i\omega_i\bar{\rho}_i' + \rho_{0i}\omega_i\alpha_i' - \alpha_i\rho_{0i}k\bar{u}_i' = 0 \qquad [38]$$

and

$$\alpha_i\rho_{0i}\omega_i\bar{u}_i' - \alpha_i k\bar{p}_i' + k\alpha_i'(\pm 1/2\alpha_i\rho_{0i}gH) = 0 \qquad [39]$$

This confirms the validity of the one-dimensional averaged conservation laws. One key difference from Ardron's solution is that we do *not* have $\bar{P}_i' = \bar{\rho}_i' c_i^2$ but, instead, from [35] and [36]

$$\bar{P}_i' + \frac{\rho_{0i}g\eta'}{2} = \bar{\rho}_i' c_i^2 \qquad [40]$$

This result could probably not have been foreseen with confidence, from ad hoc arguments. It is a consequence of the second term in [29], that plays a role in the evaluation of the mean pressure fluctuation but does not influence the mean density fluctuations, as long as we assume that there are no initial vertical density gradients.

The more realistic model in which each phase is treated as an ideal gas obeying [13] and is allowed to be both statically and dynamically compressible does satisfy $\bar{P}_j' = \bar{\rho}_j'/c_j^2$ (but only if the dynamic compression follows an isothermal path, or more generally if $(\partial p'/\partial \rho') = (\partial p_0/\partial \rho_0)$), so [40] could be regarded as resulting from an inconsistent initial set of assumptions. However, in this case, the expressions for $\bar{u}_j'$, $\bar{P}_j'$, and $\bar{\rho}_j'$ are much more complicated and, since ρ_{0j} is not uniform, we need to use some average value, $\bar{\rho}_{0j}$, in [38] and [39]. Moreover, when ρ_{0j} depends on y, we may need to be more careful in evaluating the "averages of products" that appear in the precursors of [1] and [2], particularly when deriving the perturbed equations.

Even with these insights we cannot determine the "one-dimensional" dispersion relation without using the boundary condition satisfied by the interfacial pressures. Assuming $\lambda_j h_j \ll 1$ we obtain from [32] and [33]

$$P_j^{*\prime} = -\rho_{0j} g \eta' - \rho_{0j} \frac{\omega_j^2}{\mu_j^2} \left(\frac{g}{2c_j^2} \pm \frac{1}{h_j} \right) \eta' \qquad [41]$$

which leads directly to the dispersion relation:

$$\frac{\rho_G \omega_G^2}{\mu_G^2 \alpha_G} + \frac{\rho_L \omega_L^2}{\mu_L^2 \alpha_L} = \frac{gH}{2} \left[\rho_L \left(1 + \frac{k^2}{\mu_L^2} \right) - \rho_G \left(1 + \frac{k^2}{\mu_G^2} \right) \right] \qquad [42]$$

when the two interfacial pressures are assumed equal.

Comparing [41] with [35] it follows that

$$P_j^{*\prime} = \bar{P}_j' - \rho_{0j} \frac{g}{2} \left(1 + \frac{\omega_j^2}{\mu_j^2 c_j^2} \right) \eta' \qquad [43]$$

which could not be deduced from the purely hydrostatic pressure variation assumed by Ardron.

[40] and [43] are correct to first order in η' and differ from [4] and [6]. While it may be possible *a posteriori* to discover what assumptions could have been made to derive these results from a one-dimensional approach, it is unlikely that this could have been performed *a priori* until the more comprehensive result was available for comparison.

We know of at least one set of assumptions about the one-dimensional constitutive relationships which will lead to [42] by following the procedures discussed at the beginning of this paper. However, the assumptions are made with hindsight and cannot be rigorously justified.

Ardron (1980) concluded that his work "gave confidence in the use of these standard two-fluid equations for general application.......". We are rather less optimistic since several of our results have served to illustrate the pitfalls that may be encountered on an oversimplified one-dimensional pathway of approach to what is essentially a two-dimensional problem. It is comforting, however, to note that the one-dimensional conservation laws [1] and [2] are valid for this case. The problem lies in the "effective constitutive equations" that describe the relationships between the characteristic stresses, $\bar{P}_j$ and P_j^* and the "concentrations", $\bar{\rho}_j$ and α_j; these define an effective "compressibility" for the two-fluid mixture that depends on the details of the flow pattern as well as the exciting frequency and needs to be evaluated carefully for each particular case.

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APPENDIX

Corrections to Ardron's two-dimensional solution

If we apply the method used in [10]–[26] of this paper to the two-dimensional problem addressed by Ardron (1980) in the Appendix of his paper, in which he neglects the $\hat{\rho}_j g$ term in [12] and assumes that $a_j = 0$ (uniform initial density), we obtain the dispersion relationship

$$\frac{\rho_L \omega_L^2}{\mu_L} \coth(\mu_L h_L) + \frac{\rho_G \omega_G^2}{\mu_G} \coth(\mu_G h_G) = \sigma k^2 + g(\rho_L - \rho_G) \qquad [44]$$

This differs from Ardron's [A9] and is it of interest to discover why. The explanation may be found by carefully reviewing his derivations.

The terms involving pressure in the second of his [A2] should read

$$\frac{1}{\rho_j}\left(\frac{\partial P_j}{\partial x'} - \eta_1 \frac{\partial P_j}{\partial y'}\right). \qquad [45]$$

where $\eta_1 = \partial\eta/\partial x$.

In the transformed coordinates his [A3] is, by direct substitution,

$$P_{j0} = P_0{}^* - (y' + \eta)\rho_{j0} g \qquad [46]$$

where we have introduced the notation $P_0{}^*$ to indicate that this is the pressure at the unperturbed interface.

Putting $P_j = P_{j0} + \epsilon p_j{}'$ in [45] and using [46] gives

$$\frac{1}{\rho_j}\left[-\frac{\partial \eta}{\partial x'}\rho_{j0} g + \epsilon \frac{\partial p_j{}'}{\partial x'} - \eta_1\left(-\rho_{j0} g + \epsilon \frac{\partial p_j{}'}{\partial y'}\right)\right]. \qquad [47]$$

Since $\partial\eta/\partial x'$ is the same as η_1, the term involving gravity disappears from [47] and it becomes

$$\frac{\epsilon}{\rho_j}\left(\frac{\partial p_j{}'}{\partial x'} - \eta_1 \frac{\partial p_j{}'}{\partial y'}\right) \qquad [48]$$

which is no surprise—it merely the expression of the r.h.s. of [11] in the transformed coordinate system, divided by the density. Since the second term in [48] is of second order, we conclude that the equation of motion in the x-direction is unchanged, to first order, by the coordinate transformation. In fact, we believe that to first order, the three equations represented by Ardron's [A5] should be *identical* with the similar formulation derived from [10] through [12] in the original coordinates.

For the boundary condition at the interface we have

$$P_L - P_G = -\sigma \partial^2 \eta' / \partial x^2. \qquad [49]$$

Since $P_j = P_{j0} + \epsilon p_j{}'$ and P_{j0} is equal to $P_0{}^* - \epsilon\eta'\rho_{j0} g$ at the interface, [49] becomes

$$p_L{}' - \eta'\rho_{L0} g - p_G{}' + \eta'\rho_{G0} g = \sigma \partial^2 \eta' / \partial x'^2. \qquad [50]$$

[50] is a relationship between physical variables that is independent of the coordinate system and is not changed by *any* coordinate transformation (except perhaps for the form of the curvature term). Ardron's [A6] should include the η' terms in [50], and have just the same form as one would obtain from [24] in the untransformed coordinates.

We conclude that the problem, as formulated in transformed coordinates, is identical, in both equations and boundary conditions, to the problem posed in the original coordinates, as long as only first order terms are retained.

In order to obtain an interface boundary condition and an x-direction momentum equation resembling Ardron's we must make a transformation in perturbed pressure, defining

$$\pi_j' = p_j' - g\rho_j\eta'. \qquad [51]$$

In terms of the variable π_j', [50] assumes the form of Ardron's interface condition. The x-momentum equation (the second of his [A5]) also assumes his form and the y-direction momentum equation is unchanged. However, in the continuity equation we now have to use

$$\rho_j' = \frac{p_j'}{c_j^2} = \frac{\pi_j' + g\rho_j\eta'}{c_j^2} \qquad [52]$$

so that the first of Ardron's [A5] becomes

$$\frac{1}{\rho_j c_j^2}\left(\frac{\partial \pi_j'}{\partial t} + U_{j0}\frac{\partial \pi_j'}{\partial x'}\right) + \frac{\partial u_j'}{\partial x'} + \frac{\partial v_j'}{\partial y'} + \frac{g}{c_j^2}\left(\frac{\partial \eta'}{\partial t} + U_{j0}\frac{\partial \eta'}{\partial x'}\right) = 0. \qquad [53]$$

If we follow his solution procedure we find that the first of his [A7] picks up an additional term and becomes

$$i\omega_j(\hat{\pi}_j + g\rho_j\hat{\eta}) - i\rho_j c_j^2 k\hat{u}_j = -\rho_j c_j^2\,\mathrm{d}\hat{v}_j/\mathrm{d}y \qquad [54]$$

leading to the solution

$$\hat{p}_j = B_{1j}\,e^{\mu_j y} + B_{2j}\,e^{-\mu_j y} - \rho_j\hat{\eta}g$$

and, eventually, to [44].

ADDENDUM

An assumption made by Wallis & Hutchings (1983) and by Ardron (1980) is that perturbations in density are related to perturbations in pressure by†

$$\hat{P}_j = c_j^2 \hat{\rho}_j. \tag{1}$$

This is not valid, in general, because physical property relationships apply to particles of fluid and not to a given spatial location. A particle of fluid that moves to a different elevation as a wave passes will experience a pressure change

$$\mathrm{d}P = \frac{\mathrm{d}P_0}{\mathrm{d}y}\mathrm{d}y + \mathrm{d}\hat{P} \tag{2}$$

The corresponding density change will be

$$\mathrm{d}\rho = \frac{\mathrm{d}P}{c^2} = \frac{1}{c^2}\frac{\mathrm{d}P_0}{\mathrm{d}y}\mathrm{d}y + \frac{\mathrm{d}\hat{P}}{c^2}. \tag{3}$$

Since, in general, the density change can also be expressed as

$$\mathrm{d}\rho = \frac{\mathrm{d}\rho_0}{\mathrm{d}y}\mathrm{d}y + \mathrm{d}\hat{\rho} \tag{4}$$

it is clear that [1] is only compatible with [3] and [4] if

$$\frac{\mathrm{d}P_0}{\mathrm{d}y} = c^2\frac{\mathrm{d}\rho_0}{\mathrm{d}y}. \tag{5}$$

Equation [5] is the condition for a neutrally stable atmosphere and is equivalent to requiring a particular value of a_j in [13] of Wallis & Hutchings (1982), namely

$$a_j = -g/c_j^2. \tag{6}$$

It appears that the theory is only valid for perfect gases if waves can be regarded as propagating isothermally.

Equation [6] is also the condition for gradients of pressure and density to be parallel and assures that the flow is irrotational.

Furthermore, [6] is a necessary condition for the dispersion relation [26] in Wallis & Hutchings (1982) to be equivalent to a condition of equipartition of energy. We define the following energy integrals for the perturbations over the volume of fluid in one wavelength (all evaluated to second order):

Kinetic energy

$$E_k = \int\int \frac{\rho_{j0}}{2}(\hat{u}_j^2 + \hat{v}_j^2)\,\mathrm{d}x\,\mathrm{d}y.$$

Compressive energy

$$E_c = \int\int \frac{(\hat{P}_j)^2}{2c_j^2\rho_{j0}}\,\mathrm{d}x\,\mathrm{d}y.$$

†The nomenclature of the original paper is used.

Gravitational energy

$$E_g = \frac{\pi}{2k}\eta'^2 g(\rho_L - \rho_G).$$

Surface energy

$$E_s = \sigma \int \left\{\left[1 + \left(\frac{d\eta}{dx}\right)^2\right]^{1/2} - 1\right\} dx = \sigma \frac{\pi}{2} k \eta'^2.$$

Substituting the solutions given in [31]–[35] of Wallis & Hutchings (1982) in these expressions, we find that their dispersion relation [26] is identical with

$$E_k = E_c + E_g + E_s \qquad [7]$$

for the particular case in which [6] is valid. The mean kinetic energy of the motion is equal to the sum of the mean energy "stored" by the three restoring forces.

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